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by M. E. Pittman and D. L. Waidelich

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SUMMARY

Very homogeneous fields are needed in the magnetic testing of spacecraft. One method of obtaining such fields is to use a number of circular coils on a common axis. The parameters of the best three and four coil systems were obtained by setting to zero as many terms as possible in the equation for the field along the axis of the system. The parameters are presented in the form of tables and curves.

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INTRODUCTION

In testing the magnetic properties and instruments of spacecraft, there is a need to cancel the earth's magnetic field and then produce a very homogeneous controlled magnetic field. The working volume should be conveniently accessible and the electrical design and construction should be as simple as possible. Since the working volume required is large, it appears that the use of air-core coils is the only practical solution. Most of the previous work has been on circular coils and these will be considered here, although square coils might have some constructional advantages. The volume of homogeneity for two coils (Helmholtz pair) is so small that prohibitively large coils would be needed for the required uniformity. For four coils, a much larger volume of homogeneity may be obtained for a given size of the coils. A few special solutions have been given for the four coil system but no general solution over the whole range of parameters seems to have been made. It is the purpose of this paper to present such a solution and to indicate the various optimum values. The availability of a general solution will allow the design of a system when various factors such as the size or shape of the object being tested do not allow an optimum value to be used. In preparing for the solution of the four coil system, it was found that the solution of the three coil system was useful. The general solution of the three coil system is also presented here but access to the volume of uniform field is quite limited because of the position of the center coil.

THEORY

The magnetic field of a single circular coil may be obtained by various methods, such as by the use of a scalar or vector magnetic potential (References 1 and 2). The magnetic field along the axis of the coil is

$$H = \sum_{n=0}^{\infty} a_n z^n , \tag{1}$$

where z is the distance along the axis measured from the origin O as shown in Figure 1, $z < b_1$. Also

$$a_n = \frac{NI\left(1 - x^2\right)}{2b_1^{n+1}} P'_{n+1}(x) \tag{2}$$

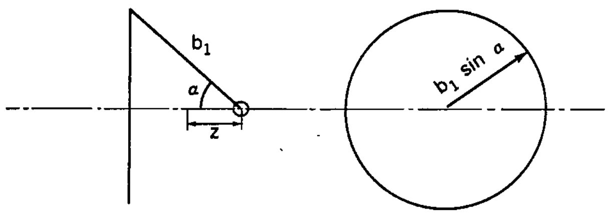

Figure 1—The single circular coil.

where N is the number of turns on the coil, I is the current through the coil, and

$$P'_{n+1}(x) = \frac{dP_{n+1}(x)}{dx} .$$

$P_n(x)$ is the nth order Legendre polynomial and

$$x = \cos\alpha .$$

Two or more of the coils on a common axis may be used to produce a more homogeneous magnetic field than is possible with one coil. This is done by making as many terms zero beyond a_0 in Equation 1 as is possible (References 3 and 4). The same results may be obtained by use of a potential (References 5 and 6).

For a symmetrical four coil system as shown in Figure 2, the magnetic field along the axis is

$$H = \sum_{n=0,2,4,\cdots}^{\infty} a_n z^n , \tag{3}$$

where

$$a_n = \frac{N_1 I_1 \left(1 - x_1^2\right)}{b_1^{\,n+1}} P'_{n+1}\left(x_1\right) + \frac{N_1 I_2 \left(1 - x_2^2\right)}{b_2^{\,n+1}} P'_{n+1}\left(x_2\right) . \tag{4}$$

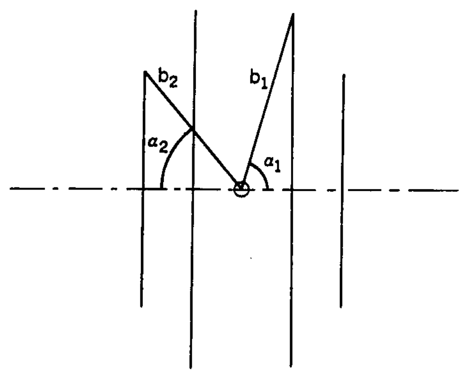

Figure 2—The four coil system.

The terms for the odd values of n in Equation 3 are not considered since they become zero because of symmetry.

THREE COIL RESULTS

The three coil solution is obtained by setting $x_1 = 0$ or $\alpha_1 = 90°$. This has the effect of making the two inner coils of Figure 2 become the one center coil of Figure 3. With three coils it is possible to make a_2 and a_4 in Equation 3 zero and then a_6 may be made a minimum. The solution is given in more detail in appendix A and results are shown in Figures 4 and 5 (A6 and B6 will be defined later). Additional results are presented in Table 1. The range of x_2 in Figures 4 and 5 is from 0.4472 to 0.7651 and the ratio of the radii $b = b_2/b_1$ ranges from infinity to zero as shown in Figure 4. This indicates that when x_2 is close to 0.4472 the diameter of the center coil should be smaller than that of the two outer coils, whereas when x_2 is close to 0.7651 the diameter of the center coil should be

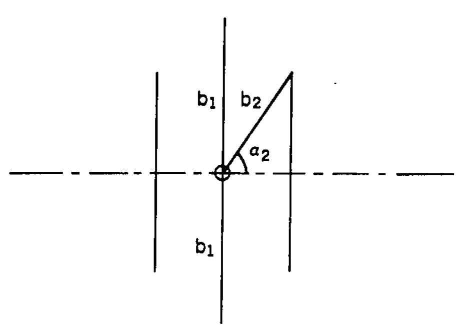

Figure 3—The three coil system.

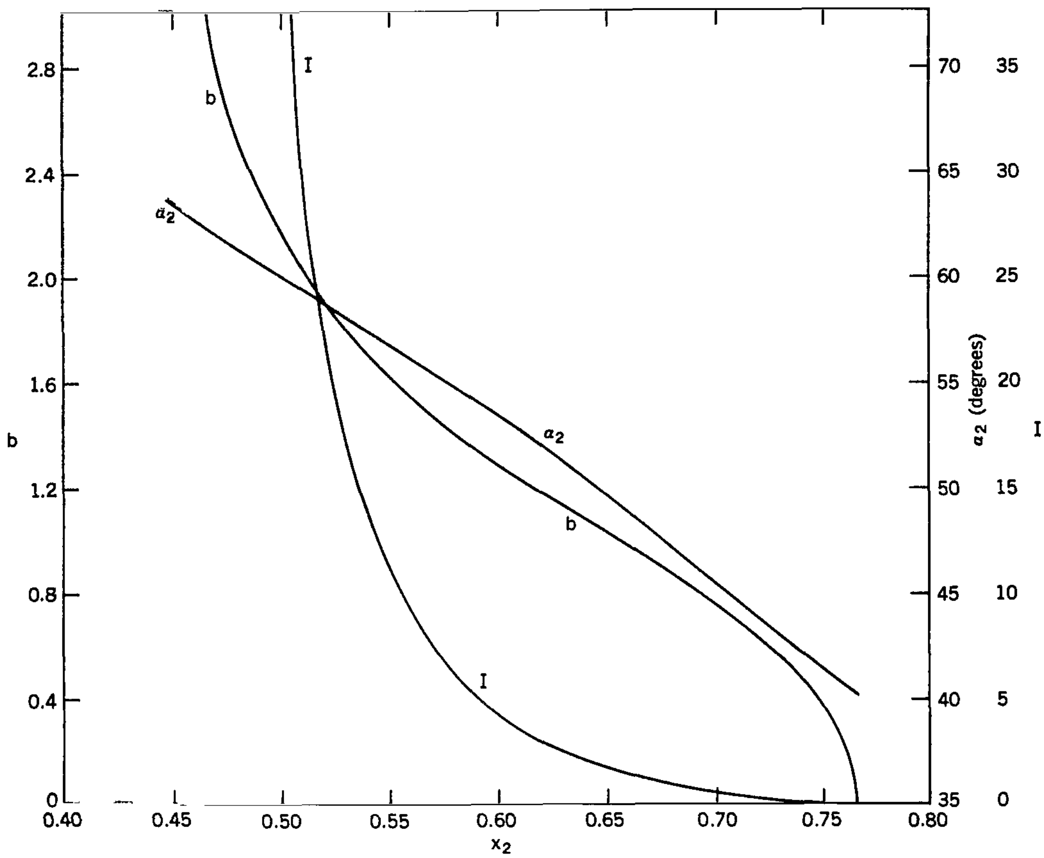

Figure 4—Calculated values of α_2, I, and b vs. x_2 for the three coil system.

larger than that of the two outer coils. From Table 1 all three coils should have equal diameters at $x_2 = 0.6051$ and at $x_2 = 0.6547$ the three coils should lie on the surface of a sphere. The range of the ratio of ampere-turns, $I = N_2I_2/N_1I_1$ is from infinity to zero as shown in Figure 4. Where $x_1 = 0$ the two inner coils of a four coil system become the center coil of a three coil system. The ampere-turns on the center coil then would be $2N_1I_1$, and the actual ratio of ampere-turns of an outer coil to the ampere-turns of the center coil is $N_2I_2/2N_1I_1 = I/2$. When x_2 is close to 0.4472, the number of ampere-turns of the center coil should be smaller than that of the outer coils, and when x_2 is close to 0.7651, the number of ampere-turns of the center coil should be larger than that of the outer coils. From Table 1 the three coils would have an equal number of ampere-turns at $x_2 = 0.6402$.

The most homogeneous field would be the one which made the remainder of the series of Equation 3 a minimum; i.e., in

$$H = a_0 \left[1 + \frac{a_6}{a_0} z^6 + \frac{a_8}{a_0} z^8 + \cdots\right] \tag{5}$$

the sum of the terms in a_6, a_8, and so on should be a minimum. Since the term containing a_6 usually is much larger than the sum of the remaining terms, making the a_6 term a minimum should give a

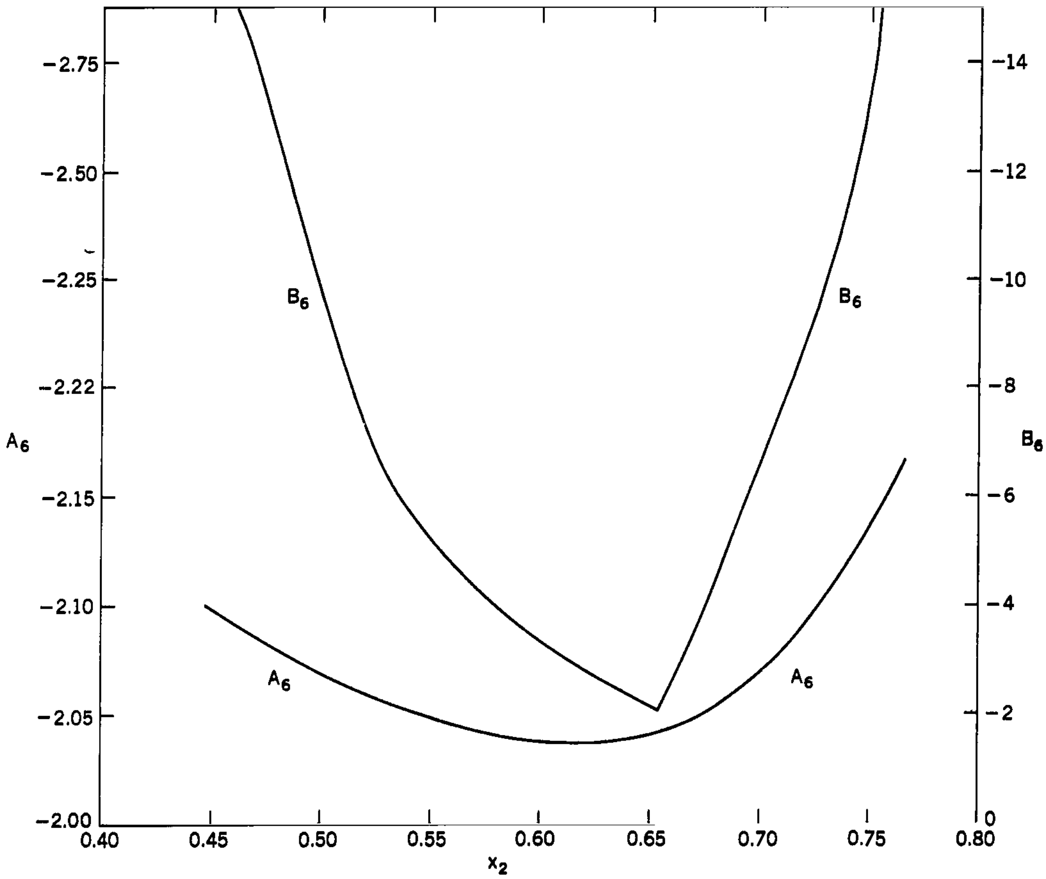

Figure 5—Calculated values of A_6 and B_6 vs. x_2 for the three coil system.

Table 1

Particular Three Coil Systems.

x_2	α_2(degrees)	b	l	A_6	B_6	Remarks
0.4472	63.42	∞	∞	-2.100	$-\infty$	End point
0.6051	52.76	1.256	3.763	-2.0369	-3.214	Coils of equal diameter, Barker's solution (Reference 3)
0.6163	51.96	1.197	3.076	-2.0364	-2.925	Minimum $\lvert A_6 \rvert$
0.6402	50.19	1.074	2.000	-2.0388	-2.351	Coils of equal ampere-turns
0.6547	49.10	1.000	1.531	-2.043	-2.043	Minimum $\lvert B_6 \rvert$, Maxwell's solution (Reference 6), coils on surface of sphere
0.7651	40.08	0.000	0.000	-2.167	$-\infty$	End point

close approximation to the optimum field. This may be done in a number of ways depending upon which parameter or combinations of parameters are assumed to remain constant. As an example let

$$A_6 = b_1^2 b_2^4 \frac{a_6}{a_0} = \bar{b}^6 \frac{a_6}{a_0} . \tag{6}$$

In Equation 6 if the mean radius from the center of the system $\bar{b} = (b_1 b_2^2)^{1/3}$ for the three coils were constant, the a_6 term would have its minimum at $x_2 = 0.6163$ as shown in Figure 5 and Table 1. As a second example let

$$B_6 = b_m^6 \frac{a_6}{a_0} = \begin{cases} b_2^6 \dfrac{a_6}{a_0} = b^2 A_6 & (b_2 \geq b_1) , \\ \\ b_1^6 \dfrac{a_6}{a_0} = b^{-4} A_6 & (b_1 \geq b_2) . \end{cases} \tag{7}$$

If the larger radius from the center of the system to the coils b_m were constant, the a_6 term would have its minimum at $x_2 = 0.6547$ as shown in Figure 5 and Table 1. If the system must be limited to a certain largest volume, probably the second optimum, that of B_6, would be the better of the two to use. It is possible to define other optima as well but it is believed that these two are the most useful and practical.

FOUR COIL RESULTS

With a four coil system such as that shown in Figure 2, it is possible to make a_2, a_4, and a_6 of Equation 3 equal zero and to make a_8 a minimum. This solution is carried out in more detail in Appendix A and the results are shown in Figures 6 and 7 and Table 2. The range of x_2 in Figures 6 and 7 is from 0.44721 to 0.87174 and the corresponding x_1 as shown in Figure 6 decreases from 0.20929 to a minimum of 0.20360 and then increases to a maximum of 0.44721. The ratio of the radii $b = b_2/b_1$ ranges from infinity to zero. Thus when x_2 is close to 0.44721, the diameter of the two inner or No. 1 coils should be smaller than that of the two outer or No. 2 coils, but when x_2 is close to 0.87174 the diameter of the No. 1 coils should be larger than that of the No. 2 coils. As indicated in Table 2 at $x_2 = 0.68519$ all four coils have the same diameter, whereas at $x_2 = 0.76505$ the four coils lie on the surface of a sphere. At 0.85363 they lie in the same plane perpendicular to the axis of the system, the No. 1 coils having a diameter 3.76797 times that of the No. 2 coils. From $x_2 = 0.85363$ to 0.87174, the No. 2 coils are closer to the center of the system than the No. 1 coils. It is interesting to notice that if the No. 1 and No. 2 coils are interchanged the two end points of Table 2 become identical. The range of the ratio of ampere-turns, $I = N_2 I_2/N_1 I_1$, goes from infinity to zero as shown in Figure 6. When x_2 is close to 0.44721 the number of ampere-turns of the No. 1 (inner) coils is smaller than that of the No. 2 (outer) coils, and for x_2 close to 0.87174 the number of ampere-turns of the No. 1 coils is larger. All coils have equal ampere-turns at $x_2 = 0.74207$.

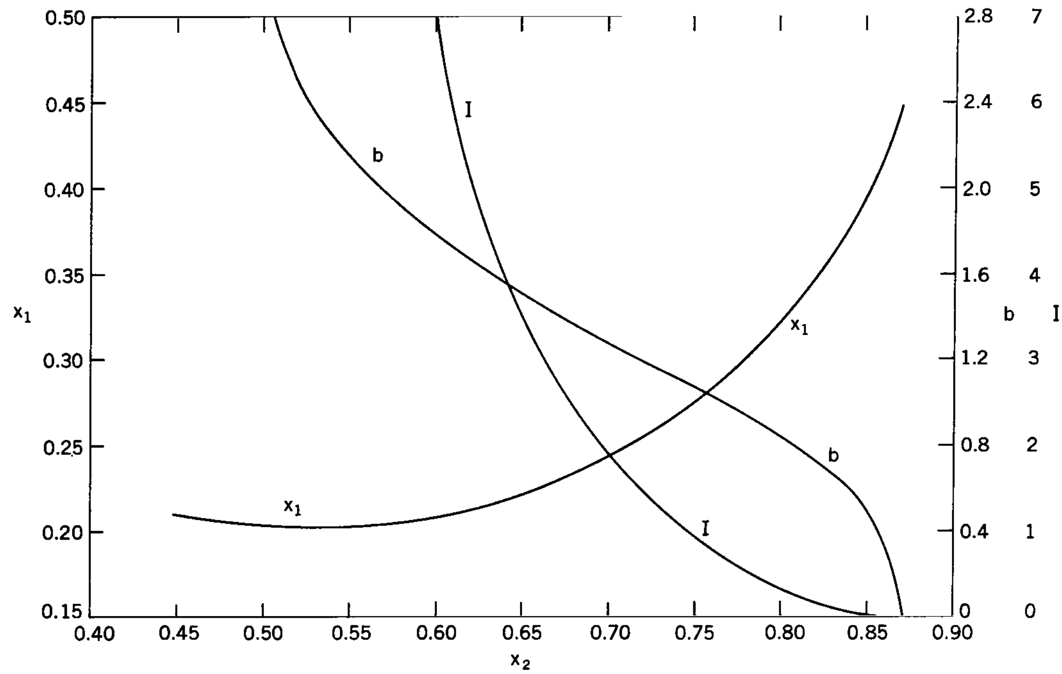

Figure 6—Calculated values of x_1, I, and b vs. x_2 for the four coil system.

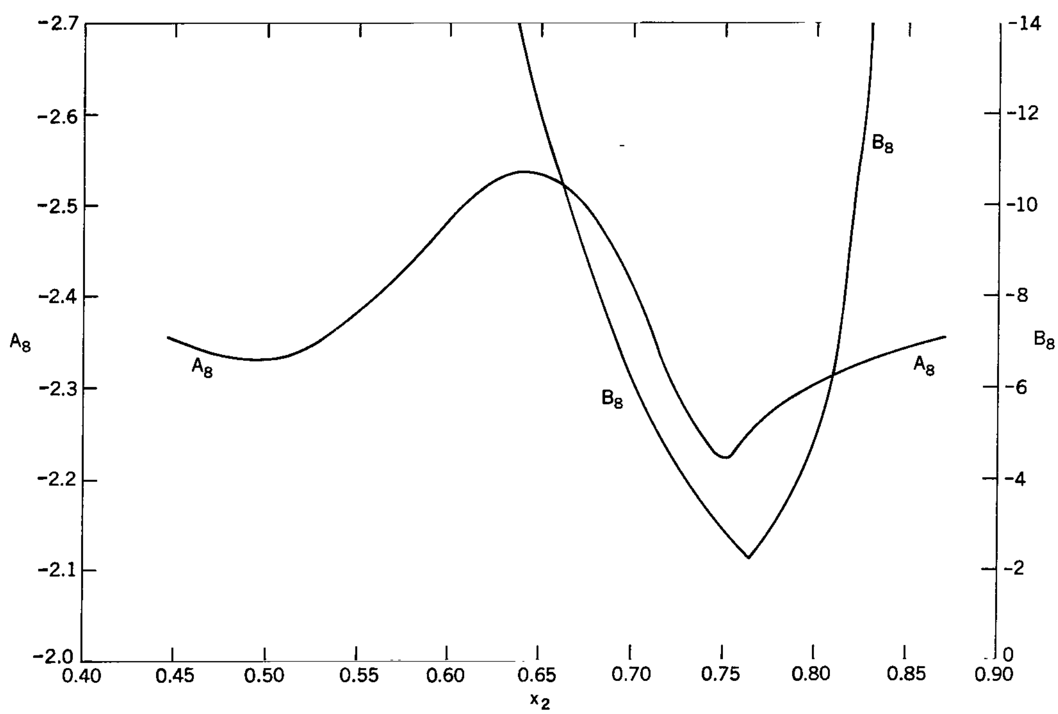

Figure 7—Calculated values of A_8 and B_8 vs. x_2 for the four coil system.

Table 2

Particular Four Coil Systems.

x_1	x_2	b	l	A_8	B_8	Remarks
0.20929	0.44721	∞	∞	-2.35384	-∞	End point
0.20360	0.51961	2.49155	46.00025	-2.34170	-90.24299	Minimum x_1
0.23629	0.68519	1.33407	2.26058	-2.47133	- 7.82798	All coils have equal diameters, Barker's solution (Reference 3)
0.26786	0.74207	1.09795	1.000	-2.23448	- 3.24721	All coils have equal ampere-turns Braunbek's solution (References 6 and 7)
0.27235	0.74842	1.07127	0.90406	-2.22196	- 2.92648	Optimum using both A_8 and A_{10}
0.27505	0.75208	1.05576	0.85165	-2.21988	- 2.75803	Optimum using A_8 only
0.28523	0.76505	1.000	0.68211	-2.25510	- 2.25510	All coils on surface of a sphere. Optimum using B_8. McKeehan's solution (References 5 and 6)
0.39864	0.85363	0.46699	0.024569	-2.34225	-49.24825	Both No. 1 and No. 2 coils lie in the same plane perpendicular to the axis of the system
0.44721	0.87174	0.000	0.000	-2.35384	-∞	End point

The most uniform field is the one which makes the remainder of the series of Equation 3 a minimum; i.e., in

$$H = a_0 \left[1 + \frac{a_8}{a_0} z^8 + \frac{a_{10}}{a_0} z^{10} + \cdots\right] , \tag{8}$$

the sum of the terms involving a_8, a_{10}, etc. should be made a minimum. A good approximation to this minimum should be that which makes the a_8 term a minimum. This may be done in a number of ways, one of which is to make the mean radius from the center of the system, $\bar{b} = (b_1 b_2)^{1/2}$, constant and define

$$A_8 = b_1^4 b_2^4 \frac{a_8}{a_0} = \bar{b}^8 \frac{a_8}{a_0} . \tag{9}$$

Figure 7 shows a curve of A_8 plotted against x_2 and Table 2 gives some values of A_8 including those at the end points. The minimum value of A_8 as given in Table 2 and shown in Figure 7 is -2.21988, which occurs at x_2 = 0.75208 and x_1 = 0.27505.

Another way of making the a_8 term a minimum is to put

$$B_8 = b_m^8 \frac{a_8}{a_0} = \begin{cases} b_2^8 \dfrac{a_8}{a_0} & (b_2 \geq b_1) \ , \\ \\ b_1^8 \dfrac{a_8}{a_0} & (b_1 \geq b_2) \ , \end{cases} \tag{10}$$

where b_m is the larger radius from the center of the system to the coils. A curve of B_8 against x_2 is shown in Figure 7 and some values of B_8 are given in Table 2. One of the values in Table 2 is the minimum value of $B_8 = -2.25510$, which occurs at $x_2 = 0.76505$ and $x_1 = 0.28523$. If more than one term is considered in the series of Equation 8, the minimum depends upon the magnitude of z. For example, consider the a_8 and a_{10} terms. They may be written as

$$\begin{aligned} \bar{b}^8 \frac{a_8}{a_0} \left(\frac{z}{\bar{b}}\right)^8 + \bar{b}^{10} \frac{a_{10}}{a_0} \left(\frac{z}{\bar{b}}\right)^{10} &= A_8 \left(\frac{z}{\bar{b}}\right)^8 + A_{10} \left(\frac{z}{\bar{b}}\right)^{10} \\ &= C \left(\frac{z}{\bar{b}}\right)^8 \ , \end{aligned} \tag{11}$$

where

$$A_{10} = \bar{b}^{10} \frac{a_{10}}{a_0} = b_1^5 b_2^5 \frac{a_{10}}{a_0} \ ,$$

$$C \equiv A_8 + A_{10} \left(\frac{z}{\bar{b}}\right)^2 \ .$$

The minimum value of C is the same as the minimum value of A_8 when $z = 0$ but when $z/\bar{b} = 0.2167$, the minimum value of C occurs at $x_2 = 0.74842$ and $x_1 = 0.27235$ as given in Table 2.

There are several other well-known four coil solutions which do not have as great a uniformity as the solutions given in Figures 6 and 7 and Table 2 because only two coefficients of Equation 3, a_2 and a_4, are made zero. Neumann (Reference 6) and Fanselau (References 4 and 8) put x_1 and x_2 at the roots of $P_5'(x) = 0$ to make a_4 zero, chose the ampere-turns of both No. 1 and No. 2 coils to be equal, and found b by putting $a_2 = 0$. Further details are given in Table 3. Fanselau (Reference 10) in another solution made a_4 zero by using the roots of $P_5'(x) = 0$. Both coils on one side of the system were put in the same plane perpendicular to the axis of the system and a_2 was made zero by choosing $I = 28.2897$. The ratio of the diameter of the smaller coil to that of the larger was 0.250495 and $b = 0.372830$. Fanselau also indicated that a similar solution could be found where both sets of coils would have the same radius. Several additional four coil solutions with both a_2 and a_4 zero are given by McKeehan (Reference 6). Scott used four coils, the inner pair having a smaller diameter than the outer pair (Reference 11). His solution had both a_2 and a_4 zero. Franzen (Reference 12) used Garrett's theory (Reference 5) to develop a theory on a finite coil cross-section for a four coil system.

Table 3

Specifications of Various Coil Systems.

Source		Ampere (Reference 6)	Helmholtz (Reference 6)	Barker (Reference 3)	Maxwell (Reference 6)	Minimum A_6
Number of Coils		1	2	3	3	3
Assumptions		None	None	Equal diameter coils	Coils on surface of sphere	None
The a's that are zero		None	a_2	a_2, a_4	a_2, a_4	a_2, a_4
Coefficient of next term, A_n		−1.5	−1.8	−2.03689	−2.0428571	−2.036426
x_1		0	0.4472136	0.0	0.0	0.0
x_2		-	-	0.605108	0.6546537	0.61627
b		-	-	1.25606	1.00	1.19697
l		-	-	3.76323	1.53125	3.075846
For an inhomogeneity of 10^{-5} (percent)	$z/\bar{b}$	0.26	4.86	13.0368	13.031	13.0373
	z/b_m	0.26	4.86	12.083	13.031	12.279

Source		Neumann-Fanselau (References 6 and 8)	Braunbek (Reference 7)	Barker (Reference 3)	McKeehan (References 5 and 6)	Minimum A_8
Number of coils		4	4	4	4	4
Assumptions		Coils have equal ampere-turns	Coils have equal ampere-turns	Coils have equal diameters	Coils on surface of sphere	None
The a's that are zero		a_2, a_4	a_2, a_4, a_6	a_2, a_4, a_6	a_2, a_4, a_6	a_2, a_4, a_6
Coefficient of next term, A_n		−1.289	−2.23448	−2.47133	−2.255102	−2.2219679
x_1		0.285232	0.26786	0.23629	0.2852315	0.2723547
x_2		0.765055	0.74207	0.68519	0.7650553	0.7484183
b		1.136009	1.09795	1.33407	1.00	1.0712777
l		1.00	1.00	2.26058	0.6821109	0.9040608
For an inhomogeneity of 10^{-5} (percent)	$z/\bar{b}$	14.07	21.672	21.480	21.607	21.677
	z/b_m	13.20	20.682	18.583	21.607	20.944

Source		Neumann-Fanselau (References 6 and 8)	Braunbek-McKeehan (References 6 and 7)	Williams-Cain (Reference 9)	Neumann-McKeehan (Reference 6)	Williams-Cain (Reference 9)
Number of coils		6	6	6	8	8
Assumptions		Coils have equal ampere-turns	Coils have equal ampere-turns	Coils lie on surface of sphere	Coils have equal ampere-turns	Coils lie on surface of sphere
The a's that are zero		a_2, a_4, a_6	a_2 to a_{10}	a_2 to a_{10}	a_2 to a_8	a_2 to a_{14}
x_1		0.20929922	0.190655	0.20929922	0.1652754	0.1652754
x_2		0.59170018	0.550274	0.59170018	0.4779250	0.4779250
x_3		0.87174003	0.843307	0.87174003	0.7387739	0.7387739
x_4		-	-	-	0.9195342	0.9195342
b_2/b_1		1.071723	1.046147	1.00	1.0222398	1.00
b_3/b_1		1.242359	1.157907	1.00	1.1827288	1.00
b_4/b_1		-	-	-	1.2382935	1.00
N_2I_2/N_1I_1		1.00	1.00	0.8270469	1.00	0.891626
N_3I_3/N_1I_1		1.00	1.00	0.5108492	1.00	0.686604
N_4I_4/N_1I_1		-	-	-	1.00	0.406992
For an inhomogeneity of 10^{-5} (percent)	z/b_m	31	34	36	16	47

In Table 3 the specifications of several coil systems are presented along with an indication of how large a sphere about the center of the system will have a given homogeneity. Several three and four coil systems are given together with the two coil system of Helmholtz and the presently known six and eight coil systems. These are compared on a basis of an inhomogeneity of 10^{-5}. An example will now be given by using the minimum A_8 solution for the four coil system:

$$\left| A_8 \left(\frac{z}{\bar{b}}\right)^8 \right| = 10^{-5} \tag{12}$$

and

$$\left(\frac{z}{\bar{b}}\right) = 0.21677 (\text{or } 21.677 \text{ percent}) .$$

This means that if $\bar{b} = 10$ feet and only the A_8 term is considered, the inhomogeneity of the magnetic field inside a sphere 2.1677 feet in radius is less than or equal to 10^{-5} or 0.001 percent of the magnetic field at the center of the system. If B_8 is used

$$\left| B_8 \left(\frac{z}{\bar{b}_m}\right)^8 \right| = 10^{-5} \tag{13}$$

and

$$\left(\frac{z}{\bar{b}_m}\right) = 0.20944 (\text{or } 20.944 \text{ percent}) .$$

The number of ampere-turns in the middle coil of the three coil system should be $2N_1 I_1$. For the six coil systems, the x's for the Neumann-Fanselau and the Williams-Cain solutions are the roots of $P_7'(x) = 0$. Also, for the eight coil systems the x's for the Neumann-McKeehan and the Williams-Cain solutions are the roots $P_9'(x) = 0$. Similar solutions with the roots of $P_{n+1}'(x) = 0$, where n is even, could be found for systems containing ten, twelve, or more coils (Reference 9). In fact Garrett mentions the fact that the solution for a sixteen coil system could be found readily (Reference 5). The Williams-Cain solution always gives the optimum solution for minimum B_{2n}.

CONCLUSIONS

Complete solutions along with tables and curves that should be useful in design work have been presented for the three and four circular-coil systems with zero for the cross-sectional areas of the coils. Comparisons of these results have been made with the two coil (Helmholtz) and the known six and eight coil systems. At the present time the following are needed:

1. A complete study of the six coil problem over the ranges of all of the parameters.
2. An investigation of the square or rectangular coil systems.
3. An analysis of the effect of finite cross-sectional area of the coils.

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Appendix A

Solution of the Coil Sytems

THREE COIL SYSTEM

Put $x_1 = 0$ in Equation 4 and let $b = b_2/b_1$ and $I = N_2I_2/N_1I_1$. Then for $n = 2$ and $n = 4$ in Equation 4

$$-\frac{3}{2}\, b^3 + I\left(1 - x_2^2\right) P_3'(x_2) = 0\,, \tag{A1}$$

$$\frac{15}{8}\, b^5 + I\left(1 - x_2^2\right) P_5'(x_2) = 0\,, \tag{A2}$$

where

$$P_3'(x_2) = \frac{3}{2}\left(5x_2^2 - 1\right)$$

and

$$P_5'(x_2) = \frac{15}{8}\left(21x_2^4 - 14x_2^2 + 1\right).$$

From Equations A1 and A2

$$\frac{I\left(1 - x_2^2\right)}{b^3} = \frac{+\frac{3}{2}}{P_3'(x_2)} = \frac{-\frac{15}{8}\,b^2}{P_5'(x_2)}\,. \tag{A3}$$

Therefore,

$$b^2 = \frac{1 - 14x_2^2 + 21x_2^4}{1 - 5x_2^2} \tag{A4}$$

$$I = \frac{b^3}{\left(1 - x_2^2\right)\left(5x_2^2 - 1\right)}\,. \tag{A5}$$

In Equation A4, b^2 must be positive, so the only possible solutions must have $0.0 \le x_2 \le 0.2852$ or $0.4472 \le x_2 \le 0.7651$ where 0.4472 is the root of $P_3'(x_2) = 0$ and 0.2852 and 0.7651 are the roots of $P_5'(x_2) = 0$. The range of values of x_2 from 0.4472 to 0.7651 produces positive values of I and the range from 0.0 to 0.2852 produces negative values of I. A negative value of I means that the current direction in the center coil is reversed from those in the two outer coils.

For the a_6 term of Equation 3 let

$$A_6 = b_1^2 b_2^4 \frac{a_6}{a_0} = b^4 b_1^6 \frac{a_6}{a_0} , \tag{A6}$$

where a_6 and a_0 are given by Equation 2. Then, by the use of Equations 4, A4, and A5

$$A_6 = -\frac{7}{16}\left(\frac{15x_2^6 - 9x_2^4 + 5x_2^2 + 5}{x_2^2 + 1}\right). \tag{A7}$$

The value of x_2 at which Equation A7 has its minimum value is 0.6163. This value of x_2 is in the range for which I is positive. Hence, for the three coil solution at least, making all currents flow in the same direction will produce a more homogeneous field. Another minimum value may be obtained by defining

$$B_6 = \begin{cases} b_2^6 \dfrac{a_6}{a_0} = b^2 A_6 & \left(0.4475 \leq x_2 \leq 0.6547\right) , \\[2ex] b_1^6 \dfrac{a_6}{a_0} = \dfrac{A_6}{b^4} & \left(0.6547 \leq x_2 \leq 0.7651\right) , \end{cases} \tag{A8}$$

where $b = 1.0$ at $x_2 = 0.6547$. Then the minimum value of B_6 occurs at $x_2 = 0.6547$.

FOUR COIL SYSTEM

For $n = 2$, 4, and 6 from Equation 4, with $b = b_2/b_1$ and $I = N_2 I_2/N_1 I_1$:

$$b^3 \left(1 - x_1^2\right) P_3'\left(x_1\right) + I\left(1 - x_2^2\right) P_3'\left(x_2\right) = 0 , \tag{A9}$$

$$b^5 \left(1 - x_1^2\right) P_5'\left(x_1\right) + I\left(1 - x_2^2\right) P_5'\left(x_2\right) = 0 , \tag{A10}$$

$$b^7 \left(1 - x_1^2\right) P_7'\left(x_1\right) + I\left(1 - x_2^2\right) P_7'\left(x_2\right) = 0 , \tag{A11}$$

where

$$P_7'\left(x_1\right) = \frac{7}{16}\left(429x_1^6 - 495x_1^4 + 135x_1^2 - 5\right) ,$$

and $P_3'\left(x_2\right)$ and $P_5'\left(x_2\right)$ are given after Equation A2. From Equations A9-A11

$$\frac{I\left(1 - x_2^2\right)}{b^3\left(1 - x_1^2\right)} = -\frac{P_3'\left(x_1\right)}{P_3'\left(x_2\right)} = -\frac{b^2 P_5'\left(x_1\right)}{P_3'\left(x_2\right)} = -\frac{b^4 P_7'\left(x_1\right)}{P_7'\left(x_2\right)} . \tag{A12}$$

Then

$$b^2 = \frac{P_3'(x_1)\,P_5'(x_2)}{P_3'(x_2)\,P_5'(x_1)} = \frac{P_5'(x_1)\,P_7'(x_2)}{P_5'(x_2)\,P_7'(x_1)} . \tag{A13}$$

From this equation:

$$\frac{P_3'(x_1)\,P_7'(x_1)}{\left[P_5'(x_1)\right]^2} = \frac{P_3'(x_2)\,P_7'(x_2)}{\left[P_5'(x_2)\right]^2} = f(x_2) \tag{A14}$$

or

$$P_3'(x_1)\,P_7'(x_1) - f(x_2)\left[P_5'(x_1)\right]^2 = 0 . \tag{A15}$$

For a given x_2, Equation A15 is solved for x_1 and the ratio of the radii b is obtained from Equation A13. The ampere-turn ratio I may be obtained by the use of Equation A12:

$$I = -\frac{b^3\left(1 - x_1^2\right)P_3'(x_1)}{\left(1 - x_2^2\right)P_3'(x_2)} . \tag{A16}$$

The use of a computer allows numerous solutions to be obtained over the range for x_2 from 0.4472 which is a root of $P_3'(x) = 0$ to 0.8717 which is a root of $P_7'(x) = 0$. The corresponding range for x_1 is from 0.2036 to 0.4472. For these values I is positive. I may be negative when both x_1 and x_2 lie in the range between 0.4472 and 0.5917 where 0.5917 is a root of $P_7'(x) = 0$. The values for negative I were not calculated because the more useful values were in the range where I is positive.

For the a_8 term of Equation 3 let

$$A_8 = b_1^4 b_2^4 \frac{a_8}{a_0} , \tag{A17}$$

where a_8 and a_0 are given by Equation 2. Then by the use of Equations 4, A13, and A16

$$A_8 = \frac{P_3'(x_1)\,P_5'(x_2)\,P_7'(x_2)\,P_9'(x_1) - P_3'(x_2)\,P_5'(x_1)\,P_7'(x_1)\,P_9'(x_2)}{P_3'(x_2)\,P_5'(x_2)\,P_7'(x_1) - P_3'(x_1)\,P_5'(x_1)\,P_7'(x_2)} , \tag{A18}$$

where

$$P_9'(x) = \frac{45}{128}\left(2431x^8 - 4004x^6 + 2002x^4 - 308x^2 + 7\right) .$$

The minimum value of $|A_8|$ is 2.21989 which occurs at $x_1 = 0.27505$ and $x_2 = 0.75209$. If the sum of both the a_8 and a_{10} terms is minimized the minimum occurs at $x_1 = 0.27235$ and $x_2 = 0.74841$. Again, another minimum may be defined by taking